EXISTENCE AND UNIQUENESS OF STRONG SOLUTIONS TO 2D g-NAVIER-STOKES EQUATIONS

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Abstract

We study the existence and uniqueness of strong solutions to the first initial boundary value problem for the two-dimensional non-autonomous $g$-Navier-Stokes equations in bounded domain.

Keywords

Strong solutions, $g$-Navier-Stokes equations, Galerkin method.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\Gamma$. In this paper we study the existence and uniqueness of strong solutions to the following two-dimensional (2D) non-autonomous $g$-Navier-Stokes equations:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= f(t) \quad \text{in} \quad (0, T) \times \Omega, \\
\nabla \cdot (gu) &= 0 \quad \text{in} \quad (0, T) \times \Omega, \\
u &= 0 \quad \text{on} \quad (0, T) \times \Gamma, \\
(1.1)
\end{align*}
\]

where $u = u(x, t) = (u_1, u_2)$ is the unknown velocity vector, $p = p(x, t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient, $u_0$ is the initial velocity.

The $g$-Navier-Stokes equations is a variation of the standard Navier-Stokes equations. More precisely, when $g \equiv \text{const}$ we get the usual Navier-Stokes equations. The 2D $g$-Navier-Stokes equations arise in a natural way when we study the standard 3D problem in thin domains.

We refer the reader to [8] for a derivation of the 2D $g$-Navier-Stokes equations from the 3D Navier-Stokes equations and a relationship between them. As mentioned in [7], good properties of the 2D $g$-Navier-Stokes equations can lead to an initiate the study of the Navier-Stokes equations on the thin three dimensional domain $\Omega_g = \Omega \times (0, g)$. In the last few years, the

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existence and asymptotic behavior of weak solutions to 2D $g$-Navier-Stokes equations have been studied extensively in both cases without and with delays (see e.g. [1], [2], [3], [5], [6], [7], [8]). However, to the best of knowledge, little seems to be known about strong solutions of the 2D $g$-Navier-Stokes equations.

In this paper, we will study the existence and uniqueness of strong solutions to the two-dimensional non-autonomous $g$-Navier-Stokes equations. To do this, we make the assumption:

\[(G) \quad g \in W^{1,\infty}(\Omega) \text{ such that:} \]

\[0 < m_0 \leq g(x) \leq M_0 \text{ for all } x = (x_1, x_2) \in \Omega, \text{ and } |\nabla g|_\infty < m_0 \lambda_1^{1/2}.\]

where $\lambda_1 > 0$ is the first eigenvalue of the $g$-Stokes operator in $\Omega$ (i.e. the operator $A$ defined in Section 2).

The rest of the paper is organized as follows. In the next section, we recall some auxiliary results on function spaces and inequalities for the nonlinear terms related to the $g$-Navier-Stokes equations. In Section 3, we prove the existence of a strong solution to the problem by using the Faedo-Galerkin method.

### 2. Preliminaries

Let $L^2(\Omega, g) = (L^2(\Omega))^2$ and $H^1_0(\Omega, g) = (H^1_0(\Omega))^2$ be endowed, respectively, with the inner products

\[(u, v)_g = \int_\Omega u.vgdx, \quad u, v \in L^2(\Omega, g),\]

and

\[((u, v))_g = \int_\Omega \sum_{j=1}^2 \nabla u_j \nabla v_j gdx, \quad u = (u_1, u_2), \quad v = (v_1, v_2) \in H^1_0(\Omega, g),\]

and norms $|u|^2 = (u, u)_g$, $||u||^2 = ((u, u))_g$. Thanks to assumption $(H1)$, the norms $|.|$ and $||.|||$ are equivalent to the usual ones in $(L^2(\Omega))^2$ and in $(H^1_0(\Omega))^2$.

Let

\[\mathcal{V} = \{u \in (C^\infty_0(\Omega))^2 : \nabla.(gu) = 0\}.\]

Denote by $H_g$ the closure of $\mathcal{V}$ in $L^2(\Omega, g)$, and by $V_g$ the closure of $\mathcal{V}$ in $H^1_0(\Omega, g)$. It follows that $V_g \subset H_g \equiv H^1_g \subset V'_g$, where the injections are dense and continuous. We will use $||.|||_*$ for the norm in $V'_g$, and $\langle ., . \rangle$ for duality pairing between $V_g$ and $V'_g$.

We now define the trilinear form $b$ by

\[b(u, v, w) = \sum_{i,j=1}^2 \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j gdx,\]
whenever the integrals make sense. It is easy to check that if \( u, v, w \in V_g \), then
\[
 b(u, v, w) = -b(u, w, v).
\]
Hence
\[
 b(u, v, v) = 0, \quad \forall u, v \in V_g.
\]
Set \( A : V_g \rightarrow V'_g \) by \( \langle Au, v \rangle = (\langle u, v \rangle)_g \), \( B : V_g \times V_g \rightarrow V'_g \) by \( \langle B(u, v), w \rangle = b(u, v, w) \).
Denote \( D(A) = \{ u \in V_g : Au \in H_g \} \), then \( D(A) = H^2(\Omega, g) \cap V_g \) and \( Au = -P_g \Delta u, \forall u \in D(A) \), where \( P_g \) is the ortho-projector from \( L^2(\Omega, g) \) onto \( H_g \).

Using the Hölder inequality, the Ladyzhenskaya inequality (when \( n = 2 \)):
\[
 |u|_{L^4} \leq c|u|^{1/2} |\nabla u|^{1/2}, \quad \forall u \in H^1_0(\Omega),
\]
and the interpolation inequalities, as in [9] one can prove the following

**Lemma 2.1.** If \( n = 2 \), then
\[
 |b(u, v, w)| \leq \begin{cases} 
 c_1|u|^{1/2} ||u||^{1/2} ||v||^{1/2} ||w||^{1/2}, & \forall u, v, w \in V_g, \\
 c_2|u|^{1/2} ||u||^{1/2} ||Av||^{1/2} ||w||^{1/2}, & \forall u \in V_g, v \in D(A), w \in H_g, \\
 c_3|u|^{1/2} |Au|^{1/2} ||v|| ||w||, & \forall u \in D(A), v \in V_g, w \in H_g, \\
 c_4|u||v|| ||w|| ||Av||^{1/2}, & \forall u \in H_g, v \in V_g, w \in D(A), 
\end{cases}
\]
(2.1)
where \( c_i, i = 1, \ldots, 4, \) are appropriate constants.

**Lemma 2.2.** Let \( u \in L^2(0, T; D(A)) \cap L^\infty(0, T; V_g) \), then the function \( Bu \) defined by
\[
 (Bu(t), v)_g = b(u(t), u(t), v), \quad \forall v \in H_g, \text{ a.e. } t \in [0, T],
\]
belongs to \( L^4(0, T; H_g) \), therefore also belongs to \( L^2(0, T; H_g) \).

**Proof.** By Lemma 2.1, for almost every \( t \in [\tau, T] \), we have
\[
 |Bu(t)| \leq c_3|u(t)|^{1/2} |Au(t)|^{1/2} ||u(t)|| \leq c_3||u(t)||^{3/2} |Au(t)|^{1/2},
\]
Then
\[
 \int_0^T |Bu(t)|^4 dt \leq c_3^4 \int_0^T ||u(t)||^6 |Au(t)|^2 dt \leq c ||u||^6_{L^\infty(0, T; V_g)} \int_0^T |Au(t)|^2 dt \leq +\infty.
\]
This completes the proof.

**Lemma 2.3.** [4] Let \( u \in L^2(0, T; V_g) \), then the function \( Cu \) defined by
\[
 (Cu(t), v)_g = (\langle \nabla g / g \cdot \nabla \rangle u, v)_g = b(\nabla g / g, u, v), \forall v \in V_g,
\]
belongs to \( L^2(0, T; H_g) \), and hence also belongs to \( L^2(0, T; V'_g) \). Moreover,
\[
 |Cu(t)| \leq \frac{|\nabla g|_\infty}{m_0} ||u(t)||, \text{ for a.e. } t \in (0, T),
\]
and
\[ ||Cu(t)||_s \leq \frac{|\nabla g|_{\infty}}{m_0 \lambda_1^{1/2}} ||u(t)||, \text{ for a.e. } t \in (0, T). \]

Since
\[ -\frac{1}{g} (\nabla g \nabla) u = -\Delta u - \left( \frac{\nabla g}{g} \right) \nabla u, \]
we have
\[ (-\Delta u, v) = ((u, v))_g + \left( \left( \frac{\nabla g}{g} \right) \nabla u, v \right)_g = (Au, v)_g + \left( \left( \frac{\nabla g}{g} \right) \nabla u, v \right)_g, \forall u, v \in V_g. \]

### 3. Existence and uniqueness of strong solutions

**Definition 3.1.** Given \( f \in L^2(0, T; H_g) \) and \( u_0 \in V_g \), a strong solution on the \((0, T)\) of problem (1.1) is a function \( u \in L^2(0, T; D(A)) \cap L^\infty(0, T; V_g) \) with \( u(0) = u_0 \), and such that
\[
\frac{d}{dt} (u(t), v)_g + \nu((u(t), v))_g + \nu(Cu(t), v)_g + b(u(t), u(t), v) = (f(t), v)_g, \quad (3.1)
\]
for all \( v \in V_g \), and for a.e. \( t \in (0, T) \).

**Remark 3.1.** From the above definition, we see that the strong solution \( u \in L^2(0, T; D(A)) \) and \( \frac{du}{dt} = f - \nu Au - Bu - Cu \in L^2(0, T; H_g) \) due to the results of Lemmas 2.2 and 2.3. By Lemma 1.2 in [15], we have \( u \in C([0, T]; V_g) \), which makes the initial condition \( u(0) = u_0 \) meaningful. It is noticed that if \( u \) is a strong solution of (1.1), then \( u \) satisfies the following energy equality:

\[ |u(t)|^2 + 2\nu \int_s^t ||u(r)||^2 dr + 2\nu \int_s^t b\left( \frac{\nabla g}{g}, u(r), u(r) \right) dr = |u(s)|^2 + 2 \int_s^t (f(r), u(r))_g dr. \]

for all \( 0 \leq s < t \leq T \).

We now prove some a priori estimates for the (sufficiently regular) strong solutions to problem (1.1).

**Lemma 3.1.** For \( u \) is strong solution of (1.1), then we have:
\[
\int_0^T ||u(t)||^2 dt \leq K_1, \quad K_1 = K_1(u_0, f, \nu, T, \lambda_1), \quad (3.2)
\]
\[
\sup_{s \in [0, T]} |u(s)|^2 \leq K_2, \quad K_2 = K_2(u_0, f, \nu, T, \lambda_1). \quad (3.3)
\]

**Proof.** From (3.1), replacing \( v \) by \( u(t) \) we get
\[
\frac{d}{dt} (u(t), u(t))_g + \nu((u(t), u(t)))_g + \nu(Cu(t), u(t))_g + b(u(t), u(t), u(t)) = (f(t), u(t))_g, \quad (3.4)
\]
Because \( b(u(t), u(t), u(t)) = 0 \) and \( (Cu(t), u(t))_g = b(\frac{\nabla g}{g}, u(t), u(t)) \), from (3.4) we have

\[
\frac{d}{dt}(u(t), u(t))_g + \nu((u(t), u(t))_g + \nu b(\frac{\nabla g}{g}, u(t), u(t)) = (f(t), u(t))_g,
\]

and therefore,

\[
\frac{d}{dt}|u(t)|^2 + 2\nu||u(t)||^2 = 2(f(t), u(t))_g - 2\nu b(\frac{\nabla g}{g}, u(t), u(t)).
\] (3.5)

Using Lemma 2.3 and the Cauchy inequality, we get

\[
\frac{d}{dt}|u(t)|^2 + 2\nu||u(t)||^2 \leq 2\epsilon\nu||u(t)||^2 + \frac{|f(t)|^2}{2\epsilon\nu\lambda_1} + 2\nu \frac{\nabla g}{\nu \lambda_1^{1/2}}||u(t)||^2,
\]

and hence

\[
\frac{d}{dt}|u(t)|^2 + 2\nu(\gamma_0 - \epsilon)||u(t)||^2 \leq \frac{|f(t)|^2}{2\epsilon\nu\lambda_1},
\] (3.6)

where \( \gamma_0 = 1 - \frac{||\nabla g||_{\infty}}{\nu \lambda_1^{1/2}} > 0 \) and \( \epsilon > 0 \) is chosen such that \( \gamma_0 - \epsilon > 0 \).

By integration in \( t \) from 0 to \( T \), after dropping unnecessary term, we obtain (3.2). Then by integration in \( t \) of (3.10) from 0 to \( s \), \( 0 < s < T \), we obtain

\[
|u(s)|^2 \leq |u_0| + \frac{1}{2\epsilon\lambda_1^{1/2}} \int_0^s |f(t)|^2 dt.
\]

Then, we have (3.3).

**Lemma 3.2.** If \( u \) is a sufficiently regular solution of (1.1), then we have:

\[
\sup_{t \in [0,T]} ||u(t)||^2 \leq K_3, \quad K_3 = K_3(K_1, K_2), \tag{3.7}
\]

\[
\int_0^T |Au(t)|^2 dt \leq K_4, \quad K_4 = K_4(K_1, K_2). \tag{3.8}
\]

**Proof.** From (3.1), replacing \( v \) by \( Au(t) \) we get

\[
\frac{d}{dt}(u(t), Au(t))_g + \nu((u(t), Au(t))_g + \nu (Cu(t), Au(t))_g + b(u(t), u(t), Au(t)) = (f(t), Au(t))_g,
\] (3.9)

Since \( ((\phi, \psi))_g = \langle A\phi, \psi \rangle \quad \forall \phi, \psi \in V_g \) This relation can be written

\[
\frac{1}{2} \frac{d}{dt}|u(t)|^2 + \nu |Au(t)|^2 + \nu (Au(t), Au(t))_g + b(u(t), u(t), Au(t)) = (f(t), Au(t))_g \leq \frac{\nu}{4} |Au(t)|^2 + \frac{1}{\nu} |f(t)|^2. \tag{3.10}
\]
Using Lemma 2.1 and 2.3, (3.10) implies
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \nu |Au(t)|^2 \\
\leq \frac{\nu}{4} |Au(t)|^2 + \frac{1}{\nu} |f(t)|^2 + c_3 |u(t)|^{1/2} |Au(t)|^{3/2} |u(t)| + \frac{\nu |\nabla g|_\infty}{m_0} |u(t)||Au(t)|. \tag{3.11}
\]

Using Young’s inequality and Cauchy’s inequality, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \nu |Au(t)|^2 \\
\leq \frac{\nu}{4} |Au(t)|^2 + \frac{1}{\nu} |f(t)|^2 \\
+ \frac{\nu}{4} |Au(t)|^2 + c_3 |u(t)|^2 |u(t)|^4 \\
+ \frac{\nu |\nabla g|_\infty}{m_0 \lambda_1^{1/2}} |Au(t)|^2 + \frac{\nu |\nabla g|_\infty \lambda_1^{1/2}}{4 m_0} |u(t)|^2. \tag{3.12}
\]

Then, we have
\[
\frac{d}{dt} \|u(t)\|^2 + \nu (1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}) |Au(t)|^2 \\
\leq 2 |f(t)|^2 + 2 c_3^4 |u(t)|^2 |u(t)|^4 + \frac{\nu |\nabla g|_\infty}{2 m_0 \lambda_1^{1/2}} |u(t)|^2. \tag{3.13}
\]

Momentarily dropping the term \( \nu (1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}) |Au(t)|^2 \), we have a differential inequality
\[
y' \leq a + \theta y
\]
where
\[
y(t) = \|u(t)\|^2, \quad a(t) = 2 |f(t)|^2, \quad \theta(t) = (2 c_3^4 |u(t)|^2 |u(t)|^2 + \frac{\nu |\nabla g|_\infty}{2 m_0 \lambda_1^{1/2}}),
\]
from which we obtain by the technique of Gronwall’s lemma:
\[
\frac{d}{dt} (y(t) \exp(- \int_0^t \theta(\tau) d\tau)) \leq a(t) \exp(- \int_0^t \theta(\tau) d\tau),
\]
\[
y(t) \leq y(0) \exp(\int_0^t \theta(\tau) d\tau) + \int_0^t a(s) \exp(\int_s^t \theta(\tau) d\tau) ds,
\]
or
\[
\|u(t)\|^2 \leq \|u_0\|^2 \exp\left( \int_0^t (2 c_3^4 |u(\tau)|^2 |u(\tau)|^2 + \frac{\nu |\nabla g|_\infty}{2 m_0 \lambda_1^{1/2}}) d\tau \right) \\
+ \frac{2}{\nu} \int_0^t |f(s)|^2 \exp\left( \int_0^t (2 c_3^4 |u(\tau)|^2 |u(\tau)|^2 + \frac{\nu |\nabla g|_\infty}{2 m_0 \lambda_1^{1/2}}) d\tau \right) ds. \tag{3.14}
\]

With Lemma 3.1 we have (3.7).

We come back to (3.13), which we integrate from 0 to \( T \), we have (3.8).
Theorem 3.1. Suppose that $f \in L^2(0,T; H_g)$ and $u_0 \in V_g$ are given. Then there exists a unique strong solution $u$ of (1.1) on $(0,T)$.

Proof. (i) Uniqueness. Let $u, v$ be two strong solutions of problem (1.1) with the same initial condition and set $w = u - v$. Then, using the energy equality, we obtain
\[
|w(t)|^2 + 2\nu \int_0^t |w(s)|^2 ds + 2\nu \int_0^t b\left(\frac{\nabla g}{g}, w(s), w(s)\right) ds = -2 \int_0^t b(w(s), v(s), w(s)) ds, \quad \forall \ t \in (0,T).
\]

By Lemma 2.1, we have
\[
|2 \int_0^t b(w(s), v(s), w(s)) ds| \leq 2c_1 \int_0^t |w(s)||w(s)||v(s)||ds \leq \nu \int_0^t |w(s)|^2 ds + \frac{c_1^2}{\nu} \int_0^t |v(s)|^2 |w(s)|^2 ds.
\]

By Lemma 2.3, we have
\[
|2\nu \int_0^t b\left(\frac{\nabla g}{g}, w(s), w(s)\right) ds| \leq 2\nu \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} \int_0^t |w(s)||w(s)| ds \leq \nu \int_0^t |w(s)|^2 ds + \frac{\nu |\nabla g|_\infty^2}{m_0^2\lambda_1} \int_0^t |w(s)|^2 ds.
\]

Thus, one has
\[
|w(t)|^2 \leq \frac{c_1^2}{\nu} \int_0^t |v(s)|^2 |w(s)|^2 ds + \frac{\nu |\nabla g|_\infty^2}{m_0^2\lambda_1} \int_0^t |w(s)|^2 ds.
\]

Hence the Gronwall’s inequality completes the proof.

(ii) Existence. We split the proof of the existence into several steps.

Step 1: A Galerkin scheme.
Let $v_1, v_2, \ldots$, be a basis of $V_g$ consisting of eigenfunctions of the operator $A$, which is orthonormal in $H_g$. Denote $V_m = \text{span}\{v_1, \ldots, v_m\}$ and consider the projector $P_m u = \sum_{j=1}^m (u, v_j) v_j$. Define also
\[
u^m(t) = \sum_{j=1}^m \alpha_{m,j}(t) v_j,
\]

where the coefficients $\alpha_{m,j}$ are required to satisfy the following system
\[
\frac{d}{dt}(u^m(t), v_j)_g + \nu(Au^m(t), v_j)_g + \nu(Cu^m(t), v_j)_g + b(u^m(t), u^m(t), v_j) = (f(t), v_j)_g, \quad \forall j = 1, \ldots, m,
\]

and the initial condition is $u^m(0) = P_m u_0$. This system of ordinary differential equations in the unknown $(\alpha_{m,1}(t), \ldots, \alpha_{m,m}(t))$ fulfills the Peano theorem, so the approximate solutions $u_m$ exist.
Step 2: A priori estimates.

From (3.15), replacing $v_j$ by $Au^m(t)$ we get

$$
\frac{d}{dt}(u^m(t), Au^m(t))_g + \nu((u^m(t), Au^m(t)))_g + \nu(Cu^m(t), Au^m(t))_g + b(u^m(t), u^m(t), Au^m(t)) = (f(t), Au^m(t))_g.
$$

(3.16)

This relation is similar to (3.9). Exactly as in Lemma 3.2, for $u^m$, with $u_0$ replace by $u^m_0$.

Since $u^m_0 = P_m u_0$ and $P_m$ is an orthogonal projector in $V_g$, then

$$
||u^m_0|| = ||P_m u_0|| \leq ||u_0||,
$$

and we then find for $u^m$ exactly the same bounds as for $u$ in (3.7) and (3.8), or

$$
uu^m \text{ remains in a bounded set of } L^2(0, T; D(A)) \cap L^\infty(0, T; V_g).
$$

(3.17)

Now, observe that (3.15) is equivalent to

$$
\frac{du^m}{dt} = -\nu Au^m - \nu Cu^m - P_m B(u^m, u^m) + P_m f(t)
$$

Hence, since Lemma 2.2 we have

$$
\{(u^m)\}' \text{ is bounded in } L^2(0, T; H_g),
$$

Step 3: Passage to the limit.

From the above estimates, we conclude that there exist $u \in L^2(0, T; D(A)) \cap L^\infty(0, T; V_g)$ with $u' \in L^2(0, T; H_g)$, and a subsequence of $\{u^m\}$, relabelled the same, such that

$$
\{u^m\} \text{ converges weakly-star to } u \text{ in } L^\infty(0, T; V_g),
$$

$$
\{u^m\} \text{ converges weak to } u \text{ in } L^2(0, T; D(A)),
$$

$$
\{(u^m)\}' \text{ converges weakly to } u' \text{ in } L^2(0, T; H_g).
$$

(3.18)

Since $\Omega$ is bounded, we can use the Compactness Lemma (see [11, Chapter III. Theorem 2.1]) to deduce the existence of a subsequence (still denoted by) $u^m$ which converges strong to $u$ in $L^2(0, T; V_g)$, and therefore also in $L^2(0, T; H_g)$.

Then we can pass to the limit in the nonlinearity $b$ thanks to the following lemma whose proof is exactly the proof of Lemma 3.2 in [9, Chapter III].

**Lemma 3.3.** If $u^m$ converges to $u u$ in $L^2(0, T; H_g)$ strongly, then for any vector function $w$ with components belong to $C^1([0, T] \times \Omega)$, we have

$$
\int_0^T b(u^m(t), u^m(t), w(t))dt \to \int_0^T b(u(t), u(t), w(t))dt.
$$


Let $\psi$ be a continuously differentiable function on $[0, T]$ with $\psi(T) = 0$. We multiply (3.15) by $\psi(t)$ and then integrate by parts. This leads to the equation:

$$\begin{align*}
- \int_0^T (u^m(t), \psi'(t))_g dt &+ \nu \int_0^T (Au^m(t), w_j \psi(t))_g dt \\
+ \nu \int_0^T (Cu^m(t), \psi(t))_g dt &+ \int_0^T b(u^m(t), u^m(t), w_j \psi(t)) dt \\
= (u^m(0), w_j)_g \psi(0) + \int_0^T (f(t), w_j \psi(t))_g dt.
\end{align*}$$

Passing to the limit, we have:

$$\begin{align*}
- \int_0^T (u(t), v\psi'(t))_g dt &+ \nu \int_0^T (Au(t), v\psi(t))_g dt \\
+ \nu \int_0^T (Cu(t), v\psi(t))_g dt &+ \int_0^T b(u(t), u(t), v\psi(t)) dt \\
= (u(0), v)_g \psi(0) + \int_0^T (f(t), v\psi(t))_g dt.
\end{align*}$$

(3.19)

holds for any $v \in V'_g$, we see that $u$ satisfies (3.1) in the distribution sense.

Finally, it remains to prove that $u$ satisfies $u(0) = 0$. For this we multiply (3.15) by $\psi(t)$, and integrate. After integrating the first term by parts, we get:

$$\begin{align*}
- \int_0^T (u(t), v\psi'(t))_g dt &+ \nu \int_0^T (Au(t), v\psi(t))_g dt \\
+ \nu \int_0^T (Cu(t), v\psi(t))_g dt &+ \int_0^T b(u(t), u(t), v\psi(t)) dt \\
= (u(0), v)_g \psi(0) + \int_0^T (f(t), v\psi(t))_g dt.
\end{align*}$$

(3.20)

By comparison with (3.19), we have:

$$(u(0) - u_0, v) \psi(0) = 0,$$

We can choose $\psi$ with $\psi(0) \neq 0$, thus $u(0) = u_0$. This completes the proof.

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