A NEW SOLUTION METHOD FOR PSEUDOMONOTONE EQUILIBRIUM PROBLEMS

Nguyen Duc Hieu\(^1\) and Bui Van Dinh\(^2\)

**Abstract**

We propose a new method for solving an equilibrium problem where the bifunction is pseudomonotone with respect to its solution set. This method can be considered as an extension of the one introduced by Solodov and Svaiter in [28] from variational inequality to equilibrium. An application to Nash-Cournot equilibrium models of electricity markets is discussed and its computational results are reported.

Bài báo đề xuất một phương pháp mới giải bài toán cân bằng với song hàm cân bằng là giả đơn điệu theo tập nghiệm của nó. Phương pháp này là một sự mở rộng của phương pháp Solodov và Svaiter (xem [28]) cho bài toán bất đẳng thức phân sang bài toán cân bằng. Đồng thời, bài báo cũng trình bày một áp dụng của phương pháp đưa ra vào việc giải bài toán cân bằng Nash-Cournot trong mô hình thị trường điện và một số kết quả tính toán số của nó.

**Index terms**

Pseudomonotone equilibria, Ky Fan inequality, auxiliary subproblem principle, projection method, Armijo linesearch, Nash-Cournot equilibrium model.

1. **Introduction**

Let \( C \) be a nonempty closed convex subset in \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) and let \( \Omega \subseteq \mathbb{R}^n \) be an open convex set containing \( C \), and \( f : \Omega \times \Omega \to \mathbb{R} \) be a bifunction such that \( f(x, x) = 0 \) for every \( x \in C \). As usual, we call such a bifunction an *equilibrium bifunction*. Consider the equilibrium problem, shortly (EP) in the sense of Blum and Oettli [3]:

\[
\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \forall y \in C. \tag{EP}
\]

This problem is also often called the *Ky Fan inequality* due to Ky Fan’s contribution to this field.

(EP) is an important subject that recently has been considered in many research papers. It is well known [3], [20] that various classes of optimization, variational inequality, Kakutani fixed point, Nash equilibria in noncooperative game theory and minimax problems can be formulated as an equilibrium problem of the form (EP). There are several solution approaches that have been developed for equilibrium problems among them the projection is one of the fundamental methods. It has been shown (see e.g. [8]) that the projection method, in

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general, is not convergent for the monotone variational inequality, which is a special case of monotone equilibrium problems. In order to obtain convergent projection algorithms, the extragradient (or double projection) algorithms have been proposed. The first extragradient method has been proposed by Korpelevich in [15] for convex optimization and saddle point problems. This method has been further extended to pseudomonotone variational inequalities and equilibrium problems [1], [8], [12], [22]. To enhance the convergence of the double projection algorithms, recently hybrid projection-cutting algorithms have been proposed for pseudomonotone inclusions and variational inequalities [27], [28].

Another fundamental approach to optimization, variational inequality and EPs is the Tikhonov regularization [30]. The Tikhonov regularization method was applied to pseudomonotone variational inequalities and equilibrium problems (see e.g. [9], [14], [16], [32] and the references therein). Unlike the monotonicity case, in this case the regularized subproblems, in general, do not inherit any monotonicity property from the original problem, and therefore the existing solution methods that require monotonicity properties cannot be directly applied to solve regularized subproblems as in the case of monotone problems. However, it has been proved (see e.g. [9]) that any Tikhonov trajectory tends to the same limit which is the projection of the starting point onto the solution set of the original pseudomonotone equilibrium problem. This result suggests that, in order to obtain the limit point in the Tikhonov regularization method for pseudomonotone equilibrium problems one can minimizing the Euclidean norm over the solution set of the original pseudomonotone equilibrium problem. The latter bilevel problem is a special case of mathematical programs with equilibrium constraints that have been considered intensively in recent years (see e.g. [6], [7], [17], [21]).

The purpose of this paper is twofold. First, we extend the projection algorithm developed by Solodov and Svaiter in [28] to equilibrium problem (EP) where the bifunction \( f \) is pseudomonotone on \( C \) with respect to its solution set. Our extension is motivated by the fact reported in [28] that this algorithm works well for pseudomonotone variational inequality problems when the projection onto the feasible domain \( C \) is computationally expensive. Next, we apply this algorithm for solving Nash-Cournot Equilibrium Models of electricity markets (see e.g. [5], [25]) and some preliminary computational results.

The paper is organized as follows. The next section contains some preliminaries on the Euclidean projection and equilibrium problems. The third section is devoted to the presentation of our algorithm and an analysis of its convergence. The last section is devoted to present an application of the proposed algorithm to Nash-cournot equilibrium models of electricity markets and its implementation.
2. Preliminaries

Throughout the paper, by $P_C$ we denote the projection operator on $C$ with the norm $\| \cdot \|$, that is

$$P_C(x) \in C : \| x - P_C(x) \| \leq \| y - x \| \quad \forall y \in C.$$ 

The following well known results on the projection operator onto a closed convex set will be used in the sequel.

Lemma 2.1. Suppose that $C$ is a nonempty closed convex set in $\mathbb{R}^n$. Then

(i) $P_C(x)$ is singleton and well defined for every $x$;

(ii) $\pi = P_C(x)$ if and only if $\langle x - \pi, y - \pi \rangle \leq 0, \forall y \in C$;

(iii) $\| P_C(x) - P_C(y) \|^2 \leq \| x - y \|^2 - \| P_C(x) - x + y - P_C(y) \|^2, \forall x, y \in C$.

We recall some well known definitions on monotonicity (see e.g. [3], [8], [13], [20], [28])

Definition 2.1. A bifunction $h : C \times C \rightarrow \mathbb{R}$ is said to be

(a) strongly monotone on $C$ with modulus $\gamma > 0$, if

$$h(x, y) + h(y, x) \leq -\gamma \| x - y \|^2 \quad \forall x, y \in C;$$

(b) monotone on $C$ if

$$h(x, y) + h(y, x) \leq 0 \quad \forall x, y \in C;$$

(c) pseudomonotone on $C$ if

$$h(x, y) \geq 0 \implies h(y, x) \leq 0 \quad \forall x, y \in C;$$

(d) pseudomonotone on $C$ with respect to $x^\ast$ if

$$h(x^\ast, y) \geq 0 \implies h(y, x^\ast) \leq 0 \quad \forall y \in C.$$ 

We say that $h$ is pseudomonotone on $C$ with respect to a set $S$ if it is pseudomonotone on $C$ with respect to every point $x^\ast \in S$.

From the definitions it follows that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\forall x^\ast \in C$.

In the sequel, we need the following assumptions

(A1) $f(., y)$ is continuous on $\Omega$ for every $y \in C$;

(A2) $f(x, .)$ is lower semicontinuous and convex on $\Omega$ for every $x \in C$;

(A3) $f$ is pseudomonotone on $C$ with respect to the solution set $S$ of (EP).
Lemma 2.2. Suppose Problem (EP) has a solution. Then under the assumptions (A1), (A2) and (A3) the solution set $S$ is closed, convex and
\[ f(x^*, y) \geq 0 \forall y \in C \text{ if and only if } f(y, x^*) \leq 0 \forall y \in C. \]

The proof of this lemma when $f$ is pseudomonotone on $C$ can be found, for instance, in [13], [20]. When $f$ is pseudomonotone with respect to the solution set of (EP), it can be done by the same way. So we omit it.

The following lemmas are well-known from the auxiliary problem principle for equilibrium problems.

**Lemma 2.3.** ([18]) Under the assumptions (A1) and (A2), a point $x^* \in C$ is a solution of (EP) if and only if it is a solution to the equilibrium problem:
\[ \text{Find } x^* \in C : f(x^*, y) + \|y - x^*\|^2 \geq 0 \forall y \in C. \] (AEP)

**Lemma 2.4.** ([18]) If bifunction $f$ satisfies the assumptions (A1), (A2), then a point $x^* \in C$ is a solution of (AEP) if and only if
\[ x^* = \text{argmin} \{f(x^*, y) + \|y - x^*\|^2 : y \in C\}. \] (CP)

Note that, since $f(x,.)$ is convex, therefore (CP) is a strongly convex program.

For each $z \in C$, by $\partial_2 f(z, z)$ we denote the subgradient of the convex function $f(z,.)$ at $z$, i.e.,
\[ \partial_2 f(z, z) := \{w \in \mathbb{R}^n : f(z, y) \geq f(z, z) + \langle w, y - z \rangle, \forall y \in C\} \]
\[ = \{w \in \mathbb{R}^n : f(z, y) \geq \langle w, y - z \rangle, \forall y \in C\}, \]
and we define the halfspace $H_z$ as
\[ H_z := \{x \in \mathbb{R}^n : \langle g, x - z \rangle \leq 0\} \] (2.1)
where $g \in \partial_2 f(z, z)$. Note that when $f(x, y) = \langle F(x), y - x \rangle$, this halfspace becomes the one introduced in [28]. The following lemma says that the hyperplane does not cut off any solution of problem (EP).

**Lemma 2.5.** Under the assumptions (A2) and (A3), one has $S \subseteq H_z$ for every $z \in C$.

**Proof.** Suppose $x^* \in S$. From $g \in \partial_2 f(z, z)$, by convexity of $f(z,.)$, it follows that
\[ \langle g, x^* - z \rangle \leq f(z, x^*) - f(z, z) \leq f(z, x^*) \forall y \in C. \]
Since $x^* \in S$ we have $f(x^*, z) \geq 0$. Then, by pseudomonotonicity of $f$ with respect to $x^*$, it follows that $f(z, x^*) \leq 0$. Thus $\langle g, x^* - z \rangle \leq 0$, which implies $x^* \in H_z$. \qed
Lemma 2.6. Under the assumptions (A1) and (A2), if \( \{z^k\} \subset C \) is a sequence such that \( \{z^k\} \) converges to \( \bar{z} \) and if \( \{g^k\} \), \( g^k \in \partial_2 f(z^k, z^k) \) for all \( k \), is a sequence converging to \( \bar{g} \), then \( \bar{g} \in \partial_2 f(\bar{z}, \bar{z}) \).

Proof. Let \( g^k \in \partial_2 (z^k, z^k) \). Then

\[
f(z^k, y) \geq f(z^k, z^k) + \langle g^k, y - z^k \rangle = \langle g^k, y - z^k \rangle \quad \forall y \in C.
\]

Taking the limit as \( k \to \infty \) on both sides of the above inequality, by the upper semicontinuity of \( f(., y) \) with respect to the first argument, we obtain

\[
f(\bar{z}, y) \geq \limsup_{k \to \infty} f(z^k, y) \geq \lim_{k \to \infty} \langle g^k, y - z^k \rangle = \langle \bar{g}, y - \bar{z} \rangle \quad \forall y \in C
\]
which, together with \( f(\bar{z}, \bar{z}) = 0 \), implies that \( \bar{g} \in \partial_2 f(\bar{z}, \bar{z}) \). \( \Box \)

We need the following lemma.

Lemma 2.7. (\([28]\)) Suppose that \( x \in C \) and \( u = P_{C \cap H_z}(x) \). Then

\[
u = P_{C \cap H_z}(\bar{x}), \text{ where } \bar{x} = P_{H_z}(x).
\]

We give here a simple proof for this lemma, which is other than that in \([28]\).

Proof. Let \( w = P_{C \cap H_z}(\bar{x}) \). We show that \( w = u \). Indeed, suppose contradiction that \( w \neq u \), then by the property of the projection onto a closed convex set, we have \( \|\bar{x} - w\| < \|\bar{x} - u\| \).

By Pythagoras’s theorem, \( \|x - u\|^2 = \|x - \bar{x}\|^2 + \|\bar{x} - u\|^2 \) and \( \|x - w\|^2 = \|x - \bar{x}\|^2 + \|\bar{x} - w\|^2 \).

Combining with \( \|x - u\| < \|x - w\| \) we obtain \( \|\bar{x} - u\| < \|\bar{x} - w\| \), which contradicts to \( \|\bar{x} - w\| < \|\bar{x} - u\| \). \( \Box \)

3. An Algorithm for Pseudomonotone Equilibrium Problems

The following algorithm can be considered as an extension of Solodov-Svaiter’s algorithm \([28]\) to (EP).

Algorithm. Pick \( x^0 \in C \) and choose two parameters \( \eta \in (0, 1) \), \( \rho > 0 \).

At each iteration \( k = 0, 1, \ldots \) having \( x^k \) do the following steps:

Step 1. Solve the strongly convex program

\[
\min \left\{ f(x^k, y) + \frac{1}{\rho}\|y - x\|^2 : \ y \in C \right\} \quad CP(x^k)
\]

to obtain its unique solution \( y^k \).

If \( y^k = x^k \), terminate: \( x^k \) is a solution of (EP). Otherwise, do Step 2.
Step 2. (Armijo linesearch rule) Find $m_k$ as the smallest positive integer number $m$ satisfying
\[
\begin{cases}
z^{k,m} = (1 - \eta^m)x^k + \eta^m y^k : \\
\langle g^{k,m}, x^k - y^k \rangle \geq \frac{1}{\rho} \|y - x\|^2 \\
\text{with } g^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m}).
\end{cases}
\tag{3.1}
\]

Step 3. Set $\eta_k := \eta^{m_k}$, $z^k := z^{k,m_k}$, $g^k := g^{k,m_k}$. Take
\[
C_k := \left\{ x \in C : \langle g^k, x - z^k \rangle \leq 0 \right\}, \quad x^{k+1} := P_{C_k}(x^k),
\tag{3.2}
\]
and go to Step 1 with $k$ is replaced by $k + 1$.

Remark 3.1. (i) $g^k \neq 0 \ \forall k$, indeed, at the begining of Step 2, $x^k \neq y^k$. By the Armijo linesearch rule we have
\[
\langle g^k, x^k - y^k \rangle \geq \frac{1}{\rho} \|y - x\|^2 > 0
\]

(ii) To implement the linesearch rule, at each iteration $k$, for a positive integer number $m$, one can check the inequality
\[
\langle g^{k,m}, x^k - y^k \rangle \geq \frac{1}{\rho} \|y - x\|^2
\]
with any $g^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m})$. If this inequality is satisfied, we are done. Otherwise, one increases $m$ by one and check again the inequality with $g^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m})$ for the new $m$. As we will show in Lemma 3.1 below that, for each iteration $k$, there exists an integer number $m > 0$ such that the inequality in the linesearch rule is satisfied for every $g^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m})$. So, to implement the linesearch rule, one needs to know only one subgradient.

Now we are going to analyze the validity and convergence of the algorithm. Our proofs are based on the proof scheme in [28] (see also [23]).

Lemma 3.1. Under the assumptions (A1), (A2), the linesearch rule (3.1) is well-defined in the sense that, at each iteration $k$, there exists an integer number $m > 0$ satisfying the inequality in (3.1) for every $g^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m})$, and if, in addition (A3) is satisfied, then for every solution $x^*$ of (EP), one has
\[
\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - \tilde{x}^k\|^2 - \left( \frac{\eta_k}{\rho \|g^k\|} \right)^2 \|x^k - y^k\|^4 \forall k,
\tag{3.3}
\]
where $\tilde{x}^k = P_{H_{x^k}}(x^k)$.

Proof. First we prove that there exists a positive integer $m_0$ such that
\[
\langle g^{k,m_0}, x^k - y^k \rangle \geq \frac{1}{\rho} \|y - x\|^2, \ \forall g^{k,m_0} \in \partial_2 f(z^{k,m_0}, z^{k,m_0}).
\]
Indeed, suppose by contradiction that, for every positive integer \( m \) and \( z^{k,m} = (1-\eta^m)x^k + \eta^my^k \) there exists \( g^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m}) \) such that

\[
\langle g^{k,m}, x^k - y^k \rangle < \frac{1}{\rho} \| y^k - x^k \|^2.
\]

Since \( z^{k,m} \to x^k \) as \( m \to \infty \), by Theorem 24.5 in [26], the sequence \( \{ g^{k,m} \}_{m=1}^{\infty} \) is bounded. Thus we may assume that \( g^{k,m} \to \bar{g} \) for some \( \bar{g} \). Taking the limit as \( m \to \infty \), from \( z^{k,m} \to x^k \) and \( g^{k,m} \to \bar{g} \), by Lemma 2.6, it follows that \( \bar{g} \in \partial_2 f(x^k, x^k) \) and

\[
\langle \bar{g}, x^k - y^k \rangle \leq \frac{1}{\rho} \| y^k - x^k \|^2.
\]

(3.4)

Since \( \bar{g} \in \partial_2 f(x^k, x^k) \), we have

\[
f(x^k, y^k) \geq f(x^k, x^k) + \langle \bar{g}, y^k - x^k \rangle = \langle \bar{g}, y^k - x^k \rangle.
\]

Combining this with (3.4) yields

\[
f(x^k, y^k) + \frac{1}{\rho} \| y^k - x^k \|^2 \geq 0,
\]

which contradicts to the fact that

\[
f(x^k, y^k) + \frac{1}{\rho} \| y^k - x^k \|^2 < 0.
\]

Therefore, the linesearch is well defined.

Now we prove (3.3). For simplicity of notation, let \( d^k := x^k - y^k \), \( H_k := H_{x^k} \). Since \( x^{k+1} = P_{C \cap H_k}(\bar{x}^k) \) and \( x^* \in S \), \( x^* \in C \cap H_k \) by Lemma 2.5. Then we have

\[
\| x^{k+1} - \bar{x}^k \|^2 \leq \langle x^* - \bar{x}^k, x^{k+1} - \bar{x}^k \rangle
\]

which together with

\[
\| x^{k+1} - x^* \|^2 = \| \bar{x}^k - x^* \|^2 + \| x^{k+1} - \bar{x}^k \|^2 + 2 \langle x^{k+1} - \bar{x}^k, \bar{x}^k - x^* \rangle
\]

implies

\[
\| x^{k+1} - x^* \|^2 \leq \| \bar{x}^k - x^* \|^2 - \| x^{k+1} - \bar{x}^k \|^2.
\]

(3.5)

Choosing

\[
\bar{x}^k = P_{H_k}(x^k) = x^k - \frac{\langle g^k, x^k - z^k \rangle}{\| g^k \|^2} g^k
\]

from (3.5) we obtain

\[
\| x^{k+1} - x^* \|^2 \leq \| x^k - x^* \|^2 - \| x^{k+1} - \bar{x}^k \|^2 - 2 \langle g^k, x^k - x^* \rangle \frac{\langle g^k, x^k - z^k \rangle}{\| g^k \|^2} + \frac{\langle g^k, x^k - z^k \rangle^2}{\| g^k \|^2}.
\]

Substituting \( x^k = z^k + \eta_k d^k \) into the last inequality we get

\[
\| x^{k+1} - x^* \|^2 \leq \| x^k - x^* \|^2 - \| x^{k+1} - \bar{x}^k \|^2 + \left( \frac{\eta_k \langle g^k, d^k \rangle}{\| g^k \|^2} \right)^2 - \frac{2\eta_k \langle g^k, d^k \rangle^2 \langle g^k, x^k - x^* \rangle}{\| g^k \|^2}.
\]
\[ = \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 - \left( \frac{\eta_k \langle g^k, d^k \rangle}{\|g^k\|} \right)^2 - \frac{2\eta_k \langle g^k, d^k \rangle}{\|g^k\|^2} \langle g^k, z^k - x^* \rangle. \]

In addition, by the Armijo linesearch rule, we have
\[ \langle g^k, x^k - y^k \rangle \geq \frac{1}{\rho} \|y^k - x^k\|^2. \]

Let \( x^* \in H_k \), we can write
\[ \|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 - \left( \frac{\eta_k}{\rho \|g^k\|} \right)^2 \|x^k - y^k\|^4 \]
as desired. \( \square \)

**Theorem 3.1.** Suppose that (EP) admits a solution and that \( f \) is jointly continuous on \( \Omega \). Then under the assumptions (A2), (A3) the sequence \( \{x^k\} \) generated by Algorithm 1 converges to a solution of (EP).

**Proof.** Let \( x^* \) be any solution of (EP). By Lemma 3.1,
\[ \|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 - \left( \frac{\eta_k}{\rho \|g^k\|} \right)^2 \|x^k - y^k\|^4, \]
which implies that the sequence \( \{\|x^k - x^*\|\} \) is nonincreasingly convergent. Thus we can deduce that the sequences \( \{x^k\}, \{y^k\} \) and \( \{z^k\} \) are bounded. Taking the limit on both sides of (3.3) we get
\[ \lim_{k \to \infty} \eta_k \|x^k - y^k\| = 0. \]  
(3.6)

We will consider two distinct cases:

**Case 1:** \( \lim \sup_{k \to \infty} \eta_k > 0 \). Then there exists \( \bar{\eta} > 0 \) and a subsequence \( \{\eta_{k_i}\} \subset \{\eta_k\} \) such that \( \eta_{k_i} > \bar{\eta} \ \forall i \), and by (3.6), one has
\[ \lim_{i \to \infty} \|x^{k_i} - y^{k_i}\| = 0. \]  
(3.7)

Since \( \{x^k\} \) is bounded, we may assume that \( x^{k_i} \) converges to some \( \bar{x} \) as \( i \to \infty \). From (3.7), \( y^{k_i} \to \bar{x} \) as \( i \to \infty \), and therefore \( z^{k_i} \to \bar{x} \). By definition of \( y^{k_i} \), we have
\[ f(x^{k_i}, y) + \frac{1}{\rho} \|y - x^{k_i}\|^2 \geq f(x^{k_i}, y^{k_i}) + \frac{1}{\rho} \|y^{k_i} - x^{k_i}\|^2 \ \forall y \in C. \]

Letting \( i \to \infty \), we obtain in the limit that
\[ f(\bar{x}, y) + \frac{1}{\rho} \|y - \bar{x}\|^2 \geq f(\bar{x}, \bar{x}) + \frac{1}{\rho} \|\bar{x} - \bar{x}\|^2 = 0. \]
Hence
\[ f(\bar{x}, y) + \frac{1}{\rho} \|y - \bar{x}\|^2 \geq 0 \ \forall y \in C, \]
which means that \( \bar{x} \) is a solution of (EP). Applying (3.3) with \( x^* = \bar{x} \) we see that the sequence \( \{\|x^k - \bar{x}\|\} \) converges. Since \( \|x^{k_i} - \bar{x}\| \to 0 \), we can conclude that the whole sequence \( \{x^k\} \) converges to \( \bar{x} \in SOL(f, C). \)
Case 2: \( \lim_{k \to \infty} \eta_k = 0 \). According to the algorithm, we have

\[
z^k = (1 - \eta_k)x^k + \eta_k y^k.
\]

As before, we may assume that the subsequence \( \{x^k_i\} \subset \{x^k\} \) converges to some point \( \bar{x} \). By the same arguments as above, we see that the sequence \( \{y^k\} \) is bounded. Thus, by taking a subsequence, if necessary, we may assume that the subsequence \( \{y^k_i\} \) converges to some point \( \bar{y} \). From the definition of \( y^k_i \), we can write

\[
f(x^k_i, y^k_i) + \frac{1}{\rho} \|y^k_i - x^k_i\|^2 \leq f(x^k_i, y) + \frac{1}{\rho} \|y - x^k_i\|^2, \quad \forall y \in C.
\]

Taking the limit as \( i \to \infty \), by the lower semicontinuity of \( f(.,.) \) and the upper semicontinuity of \( f(.,y) \) we have

\[
f(\bar{x}, \bar{y}) + \frac{1}{\rho} \|\bar{y} - \bar{x}\|^2 \leq f(\bar{x}, y) + \frac{1}{\rho} \|y - \bar{x}\|^2, \quad \forall y \in C.
\] (3.8)

In the other hand, by the Armijo linesearch rule (3.1), for \( m^i = 1 \), there exists \( g^{k_i,m^i-1} \in \partial_{2} f(z^{k_i,m^i-1}, z^{k_i,m^i-1}) \) such that

\[
\langle g^{k_i,m^i-1}, x^k_i - y^k_i \rangle < \frac{1}{\rho} \|y^k_i - x^k_i\|.
\]

Since \( z^{k_i,m^i-1} \to \bar{x} \) as \( i \to \infty \), by Theorem 24.5 in [26] we have that the sequence \( \{g^{k_i,m^i-1}\} \) is bounded. Combining this fact with Lemma 2.6 that we may assume that \( g^{k_i,m^i-1} \to \bar{g} \in \partial_{2} f(\bar{x}, \bar{x}) \), and thus the above inequality becomes

\[
\langle \bar{g}, \bar{x} - \bar{y} \rangle \leq \frac{1}{\rho} \|\bar{y} - \bar{x}\|.
\] (3.9)

From \( \bar{g} \in \partial_{2} f(\bar{x}, \bar{x}) \) it follows that \( f(\bar{x}, y) \geq f(\bar{x}, \bar{x}) + \langle \bar{g}, y - \bar{x} \rangle \) \( \forall y \in C \). In particular, \( \langle \bar{g}, \bar{x} - \bar{y} \rangle \geq -f(\bar{x}, \bar{y}) \). Combining this with (3.9) we get

\[
f(\bar{x}, \bar{y}) + \frac{1}{\rho} \|\bar{y} - \bar{x}\| \geq 0.
\] (3.10)

From (3.8) and (3.10) we have

\[
0 \leq f(\bar{x}, y) + \frac{1}{\rho} \|y - \bar{x}\| \quad \forall y \in C,
\]

which implies that \( \bar{x} \) is a solution of (EP). Now we can apply (3.3) with \( x^* = \bar{x} \), by the same arguments as above, we can conclude that the whole sequence \( \{x^k\} \) converges to \( \bar{x} \in SOL(f, C) \). \( \square \)
4. Numerical examples

In this section, we apply the proposed Algorithm to solve an equilibrium model arising from Nash-Cournot oligopolistic equilibrium problems of electricity markets. This model has been investigated in some research papers (see e.g. [5], [25]). To test the algorithm we take the example in [25]. In this example, there are \( n \) companies, each company \( i \) may possess \( I_i \) generating units. Let \( x \) denote the the vector whose entry \( x_i \) stands for the the power generating by unit \( i \). Following [5], [25] we suppose that the price \( p \) is a decreasing affine function of the \( \sigma \) with \( \sigma = \sum_{i=1}^{n^g} x_i \) where \( n^g \) is the number of all generating units, that is

\[
p(x) = 378.4 - 2 \sum_{i=1}^{n^g} x_i = p(\sigma).
\]

Then the profit made by company \( i \) is given by

\[
f_i(x) = p(\sigma) \sum_{j \in I_i} x_j - \sum_{i \in I} c_j(x_j),
\]

where \( c_j(x_j) \) is the cost for generating \( x_j \). As in [25], we suppose that the cost \( c_j(x_j) \) is given by

\[
c_j(x_j) := \max\{c^0_j(x_j), c^1_j(x_j)\}
\]

with

\[
c^0_j(x_j) := \frac{\alpha^0_j}{2} x_j^2 + \beta^0_j x_j + \gamma^0_j, \quad c^1_j(x_j) := \alpha^1_j x_j + \beta^1_j (x_j)^{1/\beta^1_j - 1} / (\beta^1_j + 1)^{1/\beta^1_j}.
\]

where \( \alpha^k_j, \beta^k_j, \gamma^k_j \) (\( k = 0, 1 \)) are given parameters.

Let \( x_j^{\text{min}} \) and \( x_j^{\text{max}} \) be the lower and upper bounds for the power generating by the unit \( j \). Then the strategy set of the model takes the form

\[
C := \{x = (x_1, \ldots, x^{n^g})^T : x_j^{\text{min}} \leq x_j \leq x_j^{\text{max}} \forall j\}.
\]

Let us introduce the vector \( q^i := (q^i_1, \ldots, q^i_{n^g}) \) with

\[
q^i_j := 1, \text{ if } j \in I_i, \text{ and } q^i_j = 0, \text{ otherwise},
\]

and then define

\[
A := 2 \sum_{i=1}^{n^c} (1 - q^i_j)(q^i)^T, \quad B := 2 \sum_{i=1}^{n^c} q^i(q^i)^T, \quad a := -387.4 \sum_{i=1}^{n^c} q^i, \quad c(x) := \sum_{j=1}^{n^g} c_j(x_j).
\]

(4.1)

(4.2)
The oligopolistic equilibrium model under consideration can be formulated by the following equilibrium problem (see [25, Lemma 7]).

\[ x^* \in C : f(x,y) = ((A + \frac{3}{2}B)x + \frac{1}{2}By + a)^T(y - x) + c(y) - c(x) \geq 0 \quad \forall y \in C. \]

We test the Algorithm for this problem with corresponds to the first model in [5] where three companies \((n^c = 3)\) are considered, and the parameters are given in the following tables.

**Table 1. The lower and upper bounds for the power generation of the generating units and companies**

<table>
<thead>
<tr>
<th>Com.</th>
<th>Gen.</th>
<th>(x^0_{\text{min}})</th>
<th>(x^0_{\text{max}})</th>
<th>(x^c_{\text{min}})</th>
<th>(x^c_{\text{max}})</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>80</td>
<td>0</td>
<td>80</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>80</td>
<td>0</td>
<td>130</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0</td>
<td>50</td>
<td>0</td>
<td>130</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0</td>
<td>55</td>
<td>0</td>
<td>125</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>0</td>
<td>30</td>
<td>0</td>
<td>125</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>0</td>
<td>40</td>
<td>0</td>
<td>125</td>
</tr>
</tbody>
</table>

**Table 2. The parameters of the generating unit cost functions**

<table>
<thead>
<tr>
<th>Gen.</th>
<th>(\alpha_j^0)</th>
<th>(\beta_j^0)</th>
<th>(\gamma_j^0)</th>
<th>(\alpha_j^1)</th>
<th>(\beta_j^1)</th>
<th>(\gamma_j^1)</th>
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<tr>
<td>1</td>
<td>0.0400</td>
<td>2.00</td>
<td>0.00</td>
<td>2.0000</td>
<td>1.0000</td>
<td>25.0000</td>
</tr>
<tr>
<td>2</td>
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<td>1.75</td>
<td>0.00</td>
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<td>1.0000</td>
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<tr>
<td>3</td>
<td>0.1250</td>
<td>1.00</td>
<td>0.00</td>
<td>1.0000</td>
<td>1.0000</td>
<td>8.0000</td>
</tr>
<tr>
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<td>3.25</td>
<td>0.00</td>
<td>3.2500</td>
<td>1.0000</td>
<td>86.2069</td>
</tr>
<tr>
<td>5</td>
<td>0.0500</td>
<td>3.00</td>
<td>0.00</td>
<td>3.0000</td>
<td>1.0000</td>
<td>20.0000</td>
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<tr>
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<td>0.00</td>
<td>3.0000</td>
<td>1.0000</td>
<td>20.0000</td>
</tr>
</tbody>
</table>

We implement the Algorithm in Matlab R2008a running on a Laptop with Intel(R) Core(TM) i3CPU M330 2.13GHz with 2GB Ram with parameter \(\tau = \frac{1}{\rho}\). To terminate the Algorithm, we use the stopping criteria \(\|x^{k+1} - x^k\| \leq \epsilon\) with a tolerance \(\epsilon = 10^{-4}\). The computational results are reported in Table 3 with some starting points and regularization parameters.
Table 3. Results obtained with some starting points and regularization parameters

<table>
<thead>
<tr>
<th>Iter(k)</th>
<th>$\tau$</th>
<th>$x_1^1$</th>
<th>$x_2^1$</th>
<th>$x_3^1$</th>
<th>$x_4^1$</th>
<th>$x_5^1$</th>
<th>$x_6^1$</th>
<th>Cpu(s)</th>
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<td>0</td>
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<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>0</td>
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<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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</tr>
<tr>
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<td>32.1428</td>
<td>15.0101</td>
<td>21.5109</td>
<td>12.6344</td>
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<td>35.9894</td>
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</tr>
<tr>
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</tr>
<tr>
<td>338</td>
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<td>15.0333</td>
<td>21.1765</td>
<td>12.8010</td>
<td>12.7992</td>
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</tr>
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<td>20</td>
<td>15</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>113</td>
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<tr>
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</table>

Table 3 shows that the number of iterations and computational time depend crucially on the regularization parameters and starting points.

5. Conclusion

We have extended a projection algorithm developed in [28] to equilibrium problems where the bifunctions are pseudomonotone with respect to the solution sets. We have tested the proposed algorithm on a Nash-Cournot oligopolistic equilibrium model of electricity markets. Some computed numerical results are reported.

Acknowledgment

The authors would like to thank the referees for their useful remarks and comments that helped us very much in revising the paper.

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