NECESSARY OPTIMALITY CONDITIONS FOR AN OPTIMAL CONTROL PROBLEM OF 2D g-NAVIER-STOKES EQUATIONS

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Abstract

Considered here is the optimal control problem of 2D g-Navier-Stokes equations. We prove the existence of optimal solutions and establish the first-order necessary optimality conditions.

Chúng tôi nghiên cứu bài toán điều khiển tối ưu của hệ phương trình g-Navier-Stokes 2 chiều. Chúng tôi chứng minh sự tồn tại nghiệm tối ưu và thiết lập các điều kiện tối ưu cần cấp một.

Index terms

g-Navier-Stokes equations; existence of optimal solution; first-order necessary optimality conditions.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary Γ. In this paper we consider the following 2D g-Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y + \nabla p & = f \text{ in } (0,T)\times\Omega, \\ \nabla\cdot(gy) & = 0 \text{ in } (0,T)\times\Omega, \\ y(0,x) & = y_0,\ x\in\Omega, \end{cases} \tag{1.1}$$

where $y = y(x,t) = (y_1, y_2)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient, y_0 is the initial velocity, the external force f plays the role of control.

The g-Navier-Stokes equations is a variation of the standard Navier-Stokes equations. More precisely, when $g \equiv$ const we get the usual Navier-Stokes equations. The 2D g-Navier-Stokes equations arise in a natural way when we study the standard 3D problem in thin domains. We refer the reader to [6] for a derivation of the 2D g-Navier-Stokes equations from the 3D Navier-Stokes equations and a relationship between them. As mentioned in [6], good properties of the 2D g-Navier-Stokes equations can lead to an

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initiate the study of the Navier-Stokes equations on the thin three dimensional domain $\Omega_g = \Omega \times (0, g)$. In the last few years, the existence and asymptotic behavior of weak solutions to 2D g-Navier-Stokes equations have been studied extensively in [1], [4], [6]. In a recent work [2], the authors proved the existence and numerical approximation of strong solutions to the 2D g-Navier-Stokes equations. The long-time behavior of the strong solutions was studied more recently in [3], [5].

In this paper we consider an optimal control with a quadratic objective functional for 2D g-Navier-Stokes equations (see Section 3 for details). To do this, we assume that the function g satisfies the following assumption:

(G) $g \in W^{1,\infty}(\Omega)$ such that:

$$0 < m_0 \leq g(x) \leq M_0 \ \forall x = (x_1, x_2) \in \Omega, \text{ and } |\nabla g|_\infty < m_0 \lambda_1^{1/2},$$

where $\lambda_1 > 0$ is the first eigenvalue of the g-Stokes operator in Ω (i.e. the operator A defined in Section 2 below).

The paper is organized as follows. In Section 2, for convenience of the reader, we recall some auxiliary results on the 2D g-Navier-Stokes equations, which will be used later. Section 3 proves the existence of optimal solutions. The first-order necessary optimality condition is proved in the last section.

2. Preliminary results

2.1. Function spaces and operators

Let $L^2(\Omega, g) = (L^2(\Omega))^2$ and $H_0^1(\Omega, g) = (H_0^1(\Omega))^2$ be endowed, respectively, with the inner products

$$(u, v)_g = \int_\Omega u \cdot v g dx, \ u, v \in L^2(\Omega, g),$$

and

$$((u, v))_g = \int_\Omega \sum_{j=1}^{2} \nabla u_j \cdot \nabla v_j g dx, \ u = (u_1, u_2), v = (v_1, v_2) \in H_0^1(\Omega, g),$$

and norms $|u|^2 = (u, u)_g$, $\|u\|^2 = ((u, u))_g$. Thanks to assumption (G), the norms $|\cdot|$ and $\|\cdot\|$ are equivalent to the usual ones in $(L^2(\Omega))^2$ and in $(H_0^1(\Omega))^2$.

Let

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^2 : \nabla \cdot (gu) = 0\}.$$

Denote by H_g the closure of $\mathcal{V}$ in $L^2(\Omega, g)$, and by V_g the closure of $\mathcal{V}$ in $H_0^1(\Omega, g)$. It follows that $V_g \subset H_g \equiv H_g' \subset V_g'$, where the injections are dense and continuous. We will use $\|\cdot\|_*$ for the norm in V_g', and $\langle \cdot, \cdot \rangle$ for duality pairing between V_g and V_g'. We refer the reader to [7], [8] for more details.

To deal with the time derivative in the state equation, we introduce the common spaces of functions y whose time derivatives y_t exists as abstract functions

$$W^{\alpha}(0,T;V_g) := \{y \in L^2(0,T;V_g) : y_t \in L^{\alpha}(0,T;V_g')\},$$
$$W(0,T) := W^2(0,T;V_g).$$

where $1 \le \alpha \le 2$. Endowed with the norms

$$\|y\|_{W^{\alpha}} := \|y\|_{W^{\alpha}(0,T;V_g)} = \|y\|_{L^2(V_g)} + \|y_t\|_{L^{\alpha}(V_g')},$$
$$\|y\|_W := \|y\|_{W^2},$$

these spaces are Banach spaces.

Set $A : V_g \to V_g'$ by $\langle Au, v\rangle = ((u,v))_g$; Denote $D(A) = \{u \in V_g : Au \in H_g\}$, then $D(A) = H^2(\Omega, g) \cap V_g$ and $Au = -P_g \Delta u \ \ \forall u \in D(A)$, where P_g is the ortho-projector from $L^2(\Omega, g)$ onto H_g.

Set $B : V_g \times V_g \to V_g'$ by $\langle B(u,v), w\rangle = b(u,v,w)$, where

$$b(u,v,w) = \sum_{j,k=1}^{2} \int_{\Omega} u_j \frac{\partial v_k}{\partial x_j} w_k g dx,$$

whenever the integrals make sense. It is easy to check that if $u, v, w \in V_g$, then

$$b(u,v,w) = -b(u,w,v).$$

Hence

$$b(u,v,v) = 0 \quad \forall u, v \in V_g.$$

Let $u \in L^2(0,T;V_g)$, then the function Cu defined by

$$(Cu(t), v)_g = ((\frac{\nabla g}{g} \cdot \nabla)u, v)_g = b(\frac{\nabla g}{g}, u, v), \forall v \in V_g,$$

belongs to $L^2(0,T;H_g)$, and hence also belongs to $L^2(0,T;V_g')$. Since

$$-\frac{1}{g}(\nabla \cdot g \nabla)u = -\Delta u - (\frac{\nabla g}{g} \cdot \nabla)u,$$

we have

$$(-\Delta u, v)_g = ((u,v))_g + ((\frac{\nabla g}{g} \cdot \nabla)u, v)_g = (Au, v)_g + ((\frac{\nabla g}{g} \cdot \nabla)u, v)_g, \quad \forall u, v \in V_g.$$

Definition 2.1 (Weak solution)**.** *Let $f \in L^2(0,T;V_g')$ and $y_0 \in H_g$ be given. A function $y \in L^2(0,T;V_g)$ with $y_t \in L^2(0,T;V_g')$, i.e. $y \in W(0,T)$, is called weak solution of problem* (1.1) *if it fulfills*

$$\begin{aligned} y_t + \nu Ay + \nu Cy + B(y) = f \quad & in\ L^2(0,T;V_g'), \\ y(0) = y_0 \quad & in\ H_g. \end{aligned} \tag{2.1}$$

The following result was proved in [1].

Theorem 2.1 (Existence and uniqueness of solutions)**.** *Let Ω be a bounded and locally Lipschitz domain in $\mathbb{R}^2$. Then for every $f \in L^2(0,T;V_g')$ and $y_0 \in H_g$, the equation* (2.1) *has a unique weak solution $y \in W(0,T)$.*

2.2. Linearized equations

We will need some results about the linearied equations. Given a state $\overline{y} \in W(0,T)$, we consider the system

$$\begin{aligned} y_t + \nu A y + \nu C y + B'(\overline{y})y = f \quad &\text{in} \;\; L^2(0,T;V_g'), \\ y(0) = y_0 \quad &\text{in} \;\; H_g. \end{aligned} \tag{2.2}$$

Here, $B'(\overline{y})y$ denotes the Fréchet derivative of B with respect to the state $\overline{y}$. It is itself a functional of $L^2(0,T;V_g')$, which for $v \in L^2(0,T;V_g)$ is given by

$$(B'(\overline{y})y, v) = \int_0^T (b(\overline{y}(t), y(t), v(t)) + b(y(t), \overline{y}(t), v(t)))dt. \tag{2.3}$$

Lemma 2.1. *[11] The operator $B : W(0,T) \to L^2(0,T;V_g')$ is twice Fréchet differentiable. All derivatives of third or higher order vanish. The first derivative is given by* (2.3). *It can be estimated as*

$$\|B'(\overline{y})y\|_{L^2(V_g')} \leq c\|\overline{y}\|_W \|y\|_W.$$

As for quadratic functions, the second derivative is independent of $\overline{y}$:

$$(B''(\overline{y})[y_1, y_2], v) = \int_0^T (b(y_1(t), y_2(t), v(t)) + b(y_2(t), y_1(t), v(t)))dt.$$

The adjoint of $B'(\overline{y})$, called $B'(\overline{y})^*$, is a linear and continuous operator from $L^2(0,T;V_g)$ to $W^*(0,T)$. It can be written as

$$(B'(\overline{y})^*\lambda, w) = \int_0^T (b(\overline{y}(t), w(t), \lambda(t)) + b(w(t), \overline{y}(t), \lambda(t)))dt.$$

Theorem 2.2. *[11] Let $f \in L^2(0,T;V_g')$, $y_0 \in H_g$ and $\overline{y} \in W(0,T)$ be given. Then the equation* (2.2) *has a unique weak solution $y \in W(0,T)$.*

2.3. The control-to-state mapping

We will study the mapping: right-hand side $\mapsto$ solution, the so-called *control-to-state mapping*. Let $u \in L^2(Q)^2$ denote the control, then we will use u as f in (1.1).

Definition 2.2 (Solution mapping)**.** *Consider the system* (1.1). *The mapping $u \mapsto y$, where y is the weak solution of* (1.1) *with the control right-hand side u and fixed initial value y_0, is denoted by S, i.e. $y = S(u)$.*

Lemma 2.2. *[11] The control-to-state mapping is Fréchet differentiable as mapping from $L^2(0,T;V_g')$ to $W(0,T)$. The derivative at $\overline{u} \in L^2(0,T;V_g')$ in direction $h \in L^2(0,T;V_g')$ is given by $S'(\overline{u})h = y$, where y is the weak solution of*

$$\begin{aligned} y_t + \nu A y + \nu C y + B'(\overline{y})y = h \quad &\textit{in } L^2(0,T;V_g'), \\ y(0) = 0 \quad &\textit{in } H_g, \end{aligned}$$

with $\overline{y} = S(\overline{u})$.

In order to establish first-order optimality conditions, we need the adjoint operator of $S'(\overline{u})$ denote by $S'(\overline{u})^*$. By Lemma 2.2, we can regard $S'(\overline{u})$ as linear operator from $L^2(0,T;V_g')$ to W_0, where W_0 is defined as a closed linear subspace of $W(0,T)$ by

$$W_0 := \{y \in W(0,T) : y(0) = 0\}.$$

Hence, the adjoint will be a mapping from W_0^* to $L^2(0,T;V_g)$.

Lemma 2.3. *[11] Let be* $\overline{u} \in L^2(Q)^2$. *Then the operator* $S'(\overline{u})^*$ *is linear and continuous from* W_0^* *to* $L^2(0,T;V_g)$. *Its action is defined as follows. Take* z *in* W_0^*, *then* $\lambda = S'(\overline{u})^* z$ *holds if and only if*

$$\langle w_t + \nu Aw + \nu Cw + B'(\overline{y})w, \lambda\rangle_{L^2(V_g'),L^2(V_g)} = \langle z, w\rangle_{W_0^*,W_0} \tag{2.4}$$

for all $w \in W_0$.

Lemma 2.4. *[11] Let be* $\overline{u} \in L^2(Q)^2$ *given. Suppose the right-hand side* z *of* (2.4) *is in the form* $z = z_1 + z_2$ *with functionals* $z_1 \in L^{4/3}(0,T;V_g') \cap W_0^*$ *and* $z_2 \in W_0^*$ *defined by* $z_2(w) = (z_T, w(T))$, $z_T \in H_g$. *Then* $\lambda = S'(\overline{u})^* z$ *is the weak solution of*

$$\begin{aligned} -\lambda_t + \nu A\lambda + \nu C\lambda + B'(\overline{y})^*\lambda &= z_1 \quad \text{in } L^{4/3}(0,T;V_g'), \\ \lambda(T) &= z_T. \end{aligned}$$

Furthermore, it holds $\lambda \in W^{4/3}(0,T)$.

3. The optimal control problem

We are considering optimal control of the instationary g-Navier-Stokes equations. The minimization of the following quadratic objective functional serves as model problem:

$$J(y,u) = \frac{\alpha_T}{2}\int_\Omega |y(x,T) - y_T(x)|^2 dx + \frac{\gamma}{2}\int_Q |u(x,t)|^2 dxdt. \tag{3.1}$$

They are weighted by the coefficients α_T, γ. The free variables-state y and control u have to fulfill the instationary 2D g-Navier-Stokes equations

$$\begin{cases} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p &= u \text{ in } Q, \\ \nabla\cdot(gy) &= 0 \text{ in } Q, \\ y &= 0 \text{ on } \Sigma, \\ y(0,x) &= y_0 \text{ in } \Omega. \end{cases} \tag{3.2}$$

The control has to satisfy inequality constraints:

$$u_{a,i}(x,t) \le u_i(x,t) \le u_{b,i}(x,t) \text{ a.e. on } Q,\ i = 1,2.$$

3.1. Setting of the problem

Let us specify the problem setting. Unless other conditions are imposed, we assume that the ingredients of the optimal problem satisfy the following:

(i) The domain Ω is supposed to be an open bounded subset of $\mathbb{R}^2$ with Lipschitz boundary Γ. We denote the time-space cylinder by $Q = \Omega \times (0, T)$, its boundary by $\Sigma = \Gamma \times (0, T)$.
(ii) The initial value y_0 is a given function in H_g. The desired state has to satisfy $y_T \in H_g$.
(iii) The coefficient α_T and the regularization parameter γ are positive real numbers.
(iv) The control constraints $u_a, u_b \in L^2(Q)^2$ have to satisfy $u_{a,i}(x,t) \le u_{b,i}(x,t)$ a.e. on Q for $i = 1, 2$, such that there exists admissible functions.

We define the set of admissible controls U_{ad} by

$$U_{ad} = \{u \in L^2(Q)^2 : u_{a,i}(x,t) \le u_i(x,t) \le u_{b,i}(x,t) \text{ a.e. on } Q, i = 1, 2\}.$$

Then U_{ad} is non-empty, convex and closed in $L^2(Q)^2$.

So we end up with the optimization problem in function space

$$\min J(y, u)$$

subject to the state equation

$$\begin{aligned} y_t + \nu Ay + \nu Cy + B(y) &= u \qquad \text{in } L^2(0, T; V_g'), \\ y(0) &= y_0 \end{aligned} \tag{3.3}$$

and the control constraint

$$u \in U_{ad}. \tag{3.4}$$

3.2. Existence of optimal solutions

We call a couple (y, u) of state and control *admissible* if it satisfies the constraints (3.3) - (3.4) of the optimal control problem. We will denote in the sequel pairs of control and state by v, e.g. $v = (y, u), \overline{v} = (\overline{y}, \overline{u})$, and so on.

First, we will show the existence of optimal solutions.

Theorem 3.1. *The optimal control problem admits a globally optimal solution $\overline{u} \in U_{ad}$ with associated state $\overline{y} \in W(0, T)$.*

Proof. The set of admissible controls is non-empty and bounded in $L^2(Q)^2$. For every control in $L^2(Q)^2$, there exists a unique solution of the state equation (3.3), see Theorem 2.1. Furthermore, the functional J is bounded from below, $J(y, u) \ge 0$. Hence, there exists the infimum of J over all admissible controls and states

$$0 \le \overline{J} = \inf_{(y,u)} J(y, u) \le \infty.$$

Moreover, there is a minimizing sequence (y_n, u_n) of admissible pairs such that $J(y_n, u_n) \to \overline{J}$ for $n \to \infty$. The set $\{u_n\}$ is bounded in $L^2(Q)^2$, then we can extract a subsequence $(y_{n'}, u_{n'})$ converging weakly in $W(0,T) \times L^2(Q)^2$ to some limit $(\overline{y}, \overline{u})$. The set of admissible controls is convex and closed in $L^2(Q)^2$, hence it is weakly closed in $L^2(Q)^2$. Thus, the control $\overline{u}$ is admissible, $\overline{u} \in U_{ad}$.

Now, we have to show that the pair $(\overline{y}, \overline{u})$ satisfies the state equation (3.3). We find for $v \in L^2(0,T;V_g)$ the convergences

$$\begin{aligned}
\langle y_{n',t}, v\rangle_{L^2(V_g'),L^2(V_g)} &\to \langle \overline{y}_t, v\rangle_{L^2(V_g'),L^2(V_g)},\\
\langle Ay_{n'}, v\rangle_{L^2(V_g'),L^2(V_g)} &\to \langle A\overline{y}, v\rangle_{L^2(V_g'),L^2(V_g)},\\
\langle Cy_{n'}, v\rangle_{L^2(V_g'),L^2(V_g)} &\to \langle C\overline{y}, v\rangle_{L^2(V_g'),L^2(V_g)},\\
\langle u_{n'}, v\rangle_{L^2(V_g'),L^2(V_g)} &\to \langle \overline{u}, v\rangle_{L^2(V_g'),L^2(V_g)}, \qquad \text{as } n' \to \infty.
\end{aligned}$$

The mapping $w \mapsto w(0)$ is linear and continuous from $W(0,T) \to H_g$. Thus, $y_n(0)$ converges weakly to $\overline{y}(0)$. We have $y_0 = y_n(0)$ for all n, hence it holds $\overline{y}(0) = y_0$. On the other hand, we have $y_{n'} \to \overline{y}$ strong in $L^2(Q)^2$, because the space $W(0,T)$ is compactly imbedded into $L^2(Q)^2$, then by using [7, Lemma 3.2], we obtain

$$\langle B(y_{n'}), v\rangle_{L^2(V_g'),L^2(V_g)} \to \langle B(\overline{y}), v\rangle_{L^2(V_g'),L^2(V_g)}, \qquad \text{for } n' \to \infty.$$

Consequently, all addends in the weak formulation of the state equation converge, and

$$\langle \overline{y}_t + \nu A\overline{y} + \nu C\overline{y} + B(\overline{y}) - \overline{u}, v\rangle_{L^2(V_g'),L^2(V_g)} = 0$$

is fulfilled for all $v \in L^2(0,T;V_g)$. Furthermore, the initial condition $\overline{y}(0) = y_0$ is satisfied. Hence, $\overline{y}$ is itself the weak solution of the state equation with right-hand side $\overline{u}$, i.e. $\overline{y} = S(\overline{u})$.

Finally, it remains to show $\overline{J} = J(\overline{y}, \overline{u})$. The objective functional consists of several norm squares, thus it is weakly lower semicontinuous which implies

$$J(\overline{y}, \overline{u}) \le \liminf J(y_n', u_n') = \overline{J}, \text{ as } n' \to \infty.$$

Since $(\overline{y}, \overline{u})$ is admissible, and $\overline{J}$ is the infimum over all admissible pairs, it follows that $\overline{J} = J(\overline{y}, \overline{u})$. The proof is complete. □

3.3. Lagrange functional

We define the Lagrange functional $\mathcal{L} : W(0,T) \times L^2(Q)^2 \times L^2(0,T;V_g) \to \mathbb{R}$ for the optimal control problem as follows:

$$\mathcal{L}(y,u,\lambda) = J(y,u) - \langle y_t + \nu Ay + \nu Cy + B(y) - u, \lambda\rangle_{L^2(V_g'),L^2(V_g)}.$$

The Lagrange function $\mathcal{L}$ is, for given $\lambda \in L^2(0,T;V_g)$, twice Fréchet-differentiable with respect to $(y,u) \in W(0,T) \times L^2(Q)^2$. The first-order derivatives of $\mathcal{L}$ with respect to y and u in direction $w \in W(0,T)$ and $h \in L^2(Q)^2$ respectively are

$$\begin{aligned}
\mathcal{L}_y(y,u,\lambda)w = {} & \alpha_T(y(T) - y_T, w(T))_{H_g}\\
& - \langle w_t + \nu Aw + \nu Cw + B'(y)w, \lambda\rangle_{L^2(V_g'),L^2(V_g)}. \qquad (3.5)
\end{aligned}$$

$$\mathcal{L}_u(y,u,\lambda)h = \gamma(u,h)_Q + (h,\lambda)_Q. \tag{3.6}$$

The second-order derivative of $\mathcal{L}$ with respect to $v = (y,u)$ in directions (w_1, h_1), $(w_2, h_2) \in W(0,T) \times L^2(Q)^2$ is

$$\mathcal{L}_{vv}(y,u,\lambda)[(w_1,h_1),(w_2,h_2)] = \mathcal{L}_{yy}(y,u,\lambda)[w_1,w_2] + \mathcal{L}_{uu}(y,u,\lambda)[h_1,h_2],$$

with

$$\mathcal{L}_{yy}(y,u,\lambda)[w_1,w_2] = \alpha_T(w_1(T),w_2(T))_{H_g} - \langle B''(y)[w_1,w_2],\lambda\rangle_{L^2(V_g'),L^2(V_g)},$$

and

$$\mathcal{L}_{uu}(y,u,\lambda)[h_1,h_2] = \gamma(h_1,h_2)_Q.$$

Theorem 3.2. *[11] Let $\lambda \in L^2(0,T;V_g)$ be given. Then the Lagrangian $\mathcal{L}$ is twice Fréchet-differentiable with respect to $v = (y,u)$ from $W(0,T) \times L^2(Q)^2$ to $\mathbb{R}$. The second-order derivative at (y,u) fulfills, together with the Lagrange multiplier λ, the estimate*

$$|\mathcal{L}_{yy}(y,u,\lambda)[w_1,w_2]| \leq c_{\mathcal{L}}(1 + \|\lambda\|)\|w_1\|\|w_2\| \quad \forall\, w_1,w_2 \in W(0,T),$$

with some constant $c_{\mathcal{L}} > 0$ that does not depend on y,u,λ,w_1,w_2.

Remark 3.1. The objective functional J is twice continuously differentiable from $W(0,T)\times L^2(Q)^2$ to $\mathbb{R}$. The reduced objective $\phi(u) := J(S(u),u)$ continuously is also twice continuous differentiable from $L^2(Q)^2$ to $\mathbb{R}$.

4. First-order necessary optimality condition

Definition 4.1. *[11] A control $\overline{u} \in U_{ad}$ is said to be locally optimal in $L^p(Q)^2$, if there exists a constant $\rho > 0$ such that*

$$J(\overline{y},\overline{u}) \leq J(y,u)$$

holds for all $u \in U_{ad}$ with $\|\overline{u} - u\|_p \leq \rho$. Here, $\overline{y}$ and y denote the states associated with $\overline{u}$ and u, e.g. $\overline{y} = S(\overline{u})$ and $y = S(u)$.

Now, we will state and prove the first-order necessary optimality condition.

Theorem 4.1. *Let $\overline{u}$ be locally optimal in $L^2(Q)^2$ with associated state $\overline{y} = S(\overline{u})$. Then there exists a unique Lagrange multiplier $\overline{\lambda} \in W^{4/3}(0,T;V_g)$, which is the weak solution of the adjoint equation*

$$\begin{aligned} -\overline{\lambda}_t + \nu A\overline{\lambda} + \nu C\overline{\lambda} + B'(\overline{y})^*\overline{\lambda} &= 0, \\ \overline{\lambda}(T) &= \alpha_T(\overline{y}(T) - y_T). \end{aligned} \tag{4.1}$$

Moreover, the variational inequality

$$(\gamma\overline{u} + \overline{\lambda}, u - \overline{u})_Q \geq 0 \;\; \forall u \in U_{ad} \tag{4.2}$$

is satisfied.

Proof. From Remark 3.1, ϕ is twice Fréchet-differentiable from $L^2(Q)^2$ to $\mathbb{R}$. By Banach space optimization principles, we know that the variational inequality

$$\phi'(\overline{u})(u-\overline{u}) \geq 0 \quad \forall u \in U_{ad} \tag{4.3}$$

is necessary for local optimality of $\overline{u}$. It remains to compute ϕ' and to derive the adjoint system. Let us write ϕ in the following way

$$\phi(u) = \frac{\alpha_T}{2}|FS(u) - y_T|_2^2 + \frac{\gamma}{2}\|u\|_2^2,$$

where F defined by $F(y) = y(T)$ is a linear and continuous operator from $W(0,T)$ to H_g. The first derivative ϕ' at $\overline{u}$ in direction $h \in L^2(Q)^2$ is characterized by

$$\phi'(\overline{u})h = \alpha_T(FS(\overline{u}) - y_T, FS'(\overline{u})h)_2 + \gamma(\overline{u}, h)_Q.$$

Now, we compute the adjoint operator of F.

First, we have by definition $\langle F^*v, w\rangle = (v, w(T))_2$ for $w \in W(0,T)$ and $v \in H_g$. This implies with $v := Fy$ that $\langle F^*Fy, w\rangle = (y(T), w(T))_2$ holds for all $y, w \in W(0,T)$. With the help of this adjoint, we transform $\phi'(\overline{u})h$ to

$$\phi'(\overline{u})h = \langle \alpha_T F^*(FS(\overline{u}) - y_T), S'(\overline{u})h\rangle + \gamma(u, h)_Q.$$

Substituting back $S(\overline{u}) = \overline{y}$, the expression

$$\phi'(\overline{u})h = \langle \alpha_T F^*(\overline{y}(T) - y_T), S'(\overline{u})h\rangle + \gamma(u, h)_Q. \tag{4.4}$$

is found. The state $\overline{y}$ is solution of the nonlinear state equation, therefore it belongs to $W(0,T)$. Altogether, the addends defining $\phi'(\overline{u})$ have the following regularity:

$$F^*(\overline{y}(T) - y_T) \in W(0,T)^*.$$

Let us define

$$\overline{\lambda} := S'(\overline{u})^*\{\alpha_T F^*(\overline{y}(T) - y_T)\}. \tag{4.5}$$

The term in brackets fits in the assumption of Lemma 2.4. Thus, we can interprete $\overline{\lambda}$ as the weak solution of

$$\begin{aligned} -\lambda_t + \nu A(\lambda) + \nu C(\lambda) + B'(\overline{y})^*\lambda = 0 \ \text{ in } L^{4/3}(0,T;V_g'), \\ \lambda(T) = \alpha_T(\overline{y}(T) - y_T). \end{aligned}$$

Furthermore, we get the regularity $\lambda \in W^{4/3}(0,T)$. The space $W^{4/3}(0,T)$ is continuously imbedded into $L^p(Q)^2$ for $p < 7/2$, see [10]. Combining the definition of $\overline{\lambda}$ in (4.5) with (4.3) and (4.4) yields

$$0 \leq \phi'(\overline{u})(u - \overline{u}) = (\gamma\overline{u} + \overline{\lambda}, u - \overline{u})_Q \ \ \forall u \in U_{ad},$$

which is the variational inequality (4.2). □

The necessary optimality conditions can be formulated equivalently using the Lagrangian.

Theorem 4.2. *Under the conditions of Theorem 4.1, it is necessary for local optimality of* $\overline{u}$ *that there exists* $\overline{\lambda} \in W^{4/3}(0,T)$ *such that*

$$\mathcal{L}_y(\overline{y},\overline{u},\overline{\lambda})w = 0 \quad \forall w \in W_0,$$
$$\mathcal{L}_y(\overline{y},\overline{u},\overline{\lambda})(u-\overline{u}) \geq 0 \quad \forall u \in U_{ad}.$$

are fulfilled.

Proof. Let $\overline{\lambda}$ be solution of (4.1), which is given by (4.5). According to (3.5), we can write $\mathcal{L}_y(\overline{y},\overline{u},\overline{\lambda})w$ as

$$\mathcal{L}_y(\overline{y},\overline{u},\overline{\lambda})w = \alpha_T(\overline{y}(T)-y_T, w(T))_{H_g} - \langle w_t + \nu Aw + \nu Cw + B'(y)w, \overline{\lambda}\rangle_{L^2(V_g'),L^2(V_g)}.$$

By construction of $\overline{\lambda}$, the right-hand side vanishes for $w \in W_0$, as Lemma 2.3. Hence, we obtain $\mathcal{L}_y(\overline{y},u,\lambda)w = 0$ for all test functions $w \in W_0$. From (3.6), the derivative of the Lagrangian with respect to the control u is given as

$$\mathcal{L}_y(\overline{y},\overline{u},\overline{\lambda})(u-\overline{u}) = (\gamma\overline{u}+\overline{\lambda}, u-\overline{u})_Q.$$

Consequently, this expression must be greater or equal to zero for all admissible control $u \in U_{ad}$. □

Remark 4.1. One may use the approach as in [9], [12] to study the second-order optimality conditions as well as the regularity and stability of optimal controls for 2D g-Navier-Stokes equations. This is the subject of our future work.

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